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Discrete Torsion in Perturbative Heterotic String Theory

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In this paper we analyze discrete torsion in perturbative heterotic string theory. In previous work we have given a purely mathematical explanation of discrete torsion as the choice of orbifold group action on a B field, in the case that $dH = 0$; in this paper we perform the analogous calculations in heterotic strings where $dH \neq 0$.

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Contents

1 Introduction

Discrete torsion is a historically-mysterious degree of freedom associated with orbifolds, originally discovered in [1]. In previous work, we explained discrete torsion for type II B fields [2, 3, 4] (as well as the M-theory three-form potential C [5]). To summarize our results, we found that

Discrete torsion is the choice of orbifold group action on the B field.

In particular, we showed that discrete torsion has nothing to do with string theory *per se*, but rather has a purely mathematical understanding.

However, in our previous work [2, 3, 4] we assumed that the curvature of the B field, namely H, satisfied the usual Bianchi identity $dH = 0$. Unfortunately this is not the case for heterotic B fields, where (as is well-known) $dH = \text{Tr } F \wedge F - \text{Tr } R \wedge R$. So, strictly speaking, the results of [2, 3, 4] do not apply to the case of the heterotic B field.

In this short paper we shall fill this gap in our understanding by examining orbifold group actions on heterotic B fields. At the end of the day, we find that the difference between any two orbifold group actions on a heterotic B field is defined by the same data as in [2, 3, 4] – so although heterotic B fields look somewhat different from type II B fields, and although orbifold group actions on heterotic B fields are twisted by comparison, the difference between any two orbifold group actions can be described the same way for heterotic B fields as for type II B fields.

We begin by working out a complete description of heterotic B-fields on local coordinate charts. Before we can accomplish that goal, however, we first review relevant facts concerning Chern-Simons forms in section 2. Once we understand Chern-Simons terms at a sufficiently deep level, we work out a local-coordinate chart description of heterotic B-fields in section 3.

Once we understand heterotic B-fields sufficiently well, we proceed to study orbifold group actions. As heterotic B fields are tied to gauge and tangent bundles, we first study orbifold group actions on principal G-bundles with connection (for general G) in section 4. (In previous work [2, 3, 4] we have exhaustively discussed principal G-bundles with connection for G abelian; here we describe the general case.) Then, we discuss the induced orbifold group actions on Chern-Simons forms in section 5. Once we have the basics down, we use the usual self-consistent bootstrap to work out orbifold group actions on heterotic B-fields in section 6.

Finally, in section 7 we conclude by discussing the differences between orbifold group actions on heterotic B-fields. We find that the differences between orbifold group actions on heterotic B fields (for fixed action on the gauge and tangent bundles) is defined by the same

data as for type II B fields [2, 3, 4], and so we recover the usual $H^2(\Gamma, U(1))$, twisted sector phases of [1], and so forth.

This paper is a continuation of the papers [4] and [5], and so readers are encouraged to read them first.

2 Review of Chern-Simons forms

Before we describe the heterotic B-field in local coordinate patches, we shall first take a moment to review Chern-Simons forms.

For simplicity, we shall assume that Tr $F \wedge F$ is normalized to be (the image of) an integral cohomology class. Assume that F is a connection on a principal G-bundle with transition functions $g_{\alpha\beta}$ (defined with respect to some good cover), and let A^α denote the connection (the gauge field) in patch U_α. On overlaps,

$$A^\alpha \;=\; g_{\alpha\beta}\, A^\beta\, g_{\alpha\beta}^{-1} \;-\; (dg_{\alpha\beta})\, g_{\alpha\beta}^{-1}$$

To set conventions, define $F = dA + A \wedge A$, then it is trivial to verify that $F^\alpha = g_{\alpha\beta} F^\beta g_{\alpha\beta}^{-1}$, and so Tr $F^\alpha \wedge F^\alpha =$ Tr $F^\beta \wedge F^\beta$.

Now, given some form that lies in the image of integral cohomology, in principle one can construct the other elements of a Čech-de Rham cocycle. The first step in this is well-known:

$$\mathrm{Tr}\; F^\alpha \wedge F^\alpha \;=\; d\,\omega_3^\alpha$$

where

$$\omega_3^\alpha \;=\; \mathrm{Tr}\; \left(A^\alpha \wedge dA^\alpha \;+\; \frac{2}{3} A^\alpha \wedge A^\alpha \wedge A^\alpha \right)$$

is the usual Chern-Simons three-form.

The second step is a little more obscure, but can also be worked out. Note that

$$\omega_3^\alpha \;-\; \omega_3^\beta \;=\; -\,\mathrm{Tr}\; \left(g_{\alpha\beta}^{-1} (dg_{\alpha\beta}) \wedge dA^\beta \right)$$

Since $g_{\alpha\beta}^{-1} dg_{\alpha\beta}$ is a closed form, and we are working on a good cover, there exists a function $\Lambda_{\alpha\beta}$ such that $g_{\alpha\beta}^{-1} dg_{\alpha\beta} = d\Lambda_{\alpha\beta}$, and so we can write

$$\omega_3^\alpha \;-\; \omega_3^\beta \;=\; d\omega_2^{\alpha\beta}$$

where

$$\omega_2^{\alpha\beta} \;=\; -\mathrm{Tr}\; \left(\Lambda_{\alpha\beta} dA^\beta \right)$$

In addition, there also exist local 1-forms $\omega_1^{\alpha\beta\gamma}$ and local functions $h_{\alpha\beta\gamma\delta}$ filling out the rest of the Čech-de Rham cocycle. We can summarize this data as follows:

$$
\begin{aligned}
\mathrm{Tr}\ F^\alpha \wedge F^\alpha &= d\omega_3^\alpha \\
\omega_3^\alpha \,-\, \omega_3^\beta &= d\omega_2^{\alpha\beta} \\
\omega_2^{\alpha\beta} \,+\, \omega_2^{\beta\gamma} \,+\, \omega_2^{\gamma\alpha} &= d\omega_1^{\alpha\beta\gamma} \\
\omega_1^{\beta\gamma\delta} \,-\, \omega_1^{\alpha\gamma\delta} \,+\, \omega_1^{\alpha\beta\delta} \,-\, \omega_1^{\alpha\beta\gamma} &= d\log h_{\alpha\beta\gamma\delta} \\
\delta h_{\alpha\beta\gamma\delta} &= 1
\end{aligned}
$$

Somewhat more formally, we have described $\mathrm{Tr}\ F \wedge F$ as the curvature of a 2-gerbe associated to the principal G-bundle with connection.

This discussion is somewhat complicated, but a simpler version also exists for $\mathrm{Tr}\ F$. We can write

$$
\begin{aligned}
\mathrm{Tr}\ F^\alpha &= d\mathrm{Tr}\ A^\alpha \\
\mathrm{Tr}\ A^\alpha \,-\, \mathrm{Tr}\ A^\beta &= -\mathrm{Tr}\ \left((dg_{\alpha\beta}) g_{\alpha\beta}^{-1} \right) \\
&= d\left(\mathrm{Tr}\ \log g_{\alpha\beta} \right) \\
&= d\left(\log\det g_{\alpha\beta} \right) \\
\delta\left(\det g_{\alpha\beta} \right) &= 1
\end{aligned}
$$

Formally, we have described $\mathrm{Tr}\ F$ as the curvature of a 0-gerbe (a principal $U(1)$ bundle with connection) associated to the principal G-bundle with connection. In fact, this associated 0-gerbe is precisely the determinant bundle.

3 Heterotic B-fields

We are now ready to discuss the B field in perturbative heterotic strings. First recall that the curvature H of the B field obeys

$$
dH \,=\, \mathrm{Tr}\ F \wedge F \,-\, \mathrm{Tr}\ R \wedge R
$$

With this in mind, to each open set U_α in a good cover, we associate a three-form H^α and a two-form B^α related as

$$
H^\alpha \,=\, dB^\alpha \,+\, \omega_{3,F}^\alpha \,-\, \omega_{3,R}^\alpha
$$

Next, how are the B fields on overlapping patches related? Recall that as part of the Green-Schwarz anomaly cancellation mechanism, gauge transformations of either the gauge bundle or the tangent bundle induce gauge transformations of B. Specifically, if under a gauge transformation

$$
\omega_{3,F}^\alpha \,\mapsto\, \omega_{3,F}^\alpha \,-\, \mathrm{Tr}\ \left(d\Lambda \wedge dA^\alpha \right)
$$

then one must simultaneously have

$$B^\alpha \mapsto B^\alpha + \mathrm{Tr}\,(\Lambda dA^\alpha)$$

so that H^α remains invariant. Since the connections on the gauge and tangent bundles on overlapping patches are related by gauge transformations (defined by the transition functions), we find that in general, the difference between two-forms B^α on overlapping patches is given by

$$B^\alpha - B^\beta = dA^{\alpha\beta} - \omega_{2,F}^{\alpha\beta} + \omega_{2,R}^{\alpha\beta} \tag{1}$$

for some local one-forms $A^{\alpha\beta}$.

Note that as a consequence of the expression above, $H^\alpha = H^\beta$ on overlapping patches, i.e., $H^\alpha = H|_{U_\alpha}$ for some globally-defined three-form H.

Next, adding the expressions (1) on each double overlap in a triple overlap, we are forced to conclude that

$$A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} = \omega_{1,F}^{\alpha\beta\gamma} - \omega_{1,R}^{\alpha\beta\gamma} + d\log h_{\alpha\beta\gamma}^B \tag{2}$$

for some $U(1)$-valued functions $h_{\alpha\beta\gamma}^B$ defined on triple overlaps.

Finally, from adding the expressions (2) on each triple overlap in a quadruple overlap, we are forced to conclude that

$$\left(h_{\beta\gamma\delta}^B\right)\left(h_{\alpha\gamma\delta}^B\right)^{-1}\left(h_{\alpha\beta\delta}^B\right)\left(h_{\alpha\beta\gamma}^B\right)^{-1} = \left(h_{\alpha\beta\gamma\delta}^F\right)^{-1}\left(h_{\alpha\beta\gamma\delta}^R\right) \tag{3}$$

Note this means that at the level of Čech cohomology, the 3-cochains $h_{\alpha\beta\gamma\delta}^F$ and $h_{\alpha\beta\gamma\delta}^R$ are cohomologous; their difference is a coboundary defined by the $h_{\alpha\beta\gamma}^B$.

To summarize, we have found that the heterotic B field is described, in local coordinate patches, by a globally-defined three-form H, local two-forms B^α, local one-forms $A^{\alpha\beta}$, and local $U(1)$-valued functions $h_{\alpha\beta\gamma}^B$ obeying

$$\begin{aligned}
dH &= \mathrm{Tr}\, F\wedge F - \mathrm{Tr}\, R\wedge R \\
H|_{U_\alpha} &= dB^\alpha + \omega_{3,F}^\alpha - \omega_{3,R}^\alpha \\
B^\alpha - B^\beta &= dA^{\alpha\beta} - \omega_{2,F}^{\alpha\beta} + \omega_{2,R}^{\alpha\beta} \\
A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} &= \omega_{1,F}^{\alpha\beta\gamma} - \omega_{1,R}^{\alpha\beta\gamma} + d\log h_{\alpha\beta\gamma}^B \\
\left(h_{\beta\gamma\delta}^B\right)\left(h_{\alpha\gamma\delta}^B\right)^{-1}\left(h_{\alpha\beta\delta}^B\right)\left(h_{\alpha\beta\gamma}^B\right)^{-1} &= \left(h_{\alpha\beta\gamma\delta}^F\right)^{-1}\left(h_{\alpha\beta\gamma\delta}^R\right)
\end{aligned}$$

More formally, the heterotic B field defines a map between the 2-gerbes with connection associated to the gauge and tangent bundles.

4 Orbifold group action on principal G-bundles

In prior work [2, 4] we have exhaustively discussed orbifold group actions on principal $U(1)$ bundles. In order to discuss orbifold group actions in heterotic string theory, however, we need to examine orbifold group actions on principal G-bundles for more general Lie groups G.

To set conventions, assume we have a bundle with connection described by $\mathrm{Ad}(G)$-valued gauge fields A^α (one for each element U_α of a "good invariant" cover, as described in [2, 4]) and transition functions $g_{\alpha\beta}$, obeying

$$\begin{aligned} A^\alpha &= g_{\alpha\beta}\, A^\beta\, g_{\alpha\beta}^{-1} \;-\; (d g_{\alpha\beta})\, g_{\alpha\beta}^{-1} \\ g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} &= 1 \end{aligned}$$

Proceeding as in [2, 3, 4], define Čech cochains γ^g_α by

$$g^* g_{\alpha\beta} \;=\; (\gamma^g_\alpha)\; (g_{\alpha\beta})\; \left(\gamma^g_\beta\right)^{-1} \tag{4}$$

From expanding $(g_1 g_2)^* g_{\alpha\beta}$ in two different ways, we are led to demand

$$\gamma^{g_1 g_2}_\alpha \;=\; (g_2^* \gamma^{g_1}_\alpha)\; (\gamma^{g_2}_\alpha) \tag{5}$$

and from demanding consistency of A^α on overlaps, we are led to derive (as in [4])

$$g^* A^\alpha \;=\; (\gamma^g_\alpha)\; A^\alpha\; (\gamma^g_\alpha)^{-1} \;+\; (\gamma^g_\alpha)\; d\, (\gamma^g_\alpha)^{-1} \tag{6}$$

Now, in [2, 3, 4] we pointed out that both orbifold $U(1)$ Wilson lines and discrete torsion arise as the differences between orbifold group actions. (Put another way, the set of orbifold group actions is only a set in general, not a group, but it is acted upon by a group in those cases.) Let us attempt to repeat that analysis here. Let (γ^g_α), $(\overline{\gamma}^g_\alpha)$ define a pair of orbifold group actions on some principal G-bundle with connection, as above. Define

$$\phi^g_\alpha \;=\; (\overline{\gamma}^g_\alpha)^{-1}\; (\gamma^g_\alpha) \tag{7}$$

By expressing $g^* g_{\alpha\beta}$ in terms of these two actions, we find

$$\phi^g_\alpha\, g_{\alpha\beta} \;=\; g_{\alpha\beta}\, \phi^g_\beta \tag{8}$$

The expression above for (ϕ^g_α) shows that (ϕ^g_α) defines a base-preserving automorphism of the principal G-bundle [6, section 5.5]. Base-preserving automorphisms of a principal G-bundle are gauge transformations, so this means that (ϕ^g_α) defines a gauge transformation of the bundle.

The reader will probably be slightly confused to hear that equation (8) implies that (ϕ^g_α) defines a gauge transformation. After all, one usually thinks of gauge transformations of bundles as being global maps into G, and if (ϕ^g_α) defines a global map, then one would expect that $\phi^g_\alpha = \phi^g_\beta$ on $U_\alpha \cap U_\beta$, not equation (8). Unfortunately, working at the level of Čech cochains means implicitly working in local trivializations, and for general G, including local trivializations makes the relationship between bundle automorphisms and gauge transformations less transparent.

So far we have argued that the difference between any two orbifold group actions on a principal G bundle is defined by a set of gauge transformations. This is very reminiscent of [2, 3, 4] where we argued that the difference between any two orbifold group actions on a principal $U(1)$ bundle or on a B field is defined by a set of gauge transformations. However, there is an important difference in the present case – although the difference between any two orbifold group actions is a set of gauge transformations, the gauge transformations do not form a representation of the orbifold group in general.

Specifically, from equation (5) we find that

$$\phi^{g_1 g_2}_\alpha \;=\; \left(\overline{\gamma}^{g_2}_\alpha\right)^{-1} \left(g_2^* \phi^{g_1}_\alpha\right) \left(\gamma^{g_2}_\alpha\right) \tag{9}$$

In order for the gauge transformations ϕ^g_α to define a representation of the orbifold group, we would have needed $\phi^{g_1 g_2}_\alpha = (g_2^* \phi^{g_1}_\alpha)(\phi^{g_2}_\alpha)$, but we see that this will only be true if $\overline{\gamma}^{g_2}_\alpha$ commutes with $g_2^* \phi^{g_1}_\alpha$, which will not be true in general.

However, in very special cases one can sometimes still recover a description of orbifold Wilson lines for principal G-bundles with connection in terms of $\mathrm{Hom}(\Gamma, G)/G$, the description most familiar to physicists. Specialize to the canonically trivial bundle (i.e., $g_{\alpha\beta} \equiv 1$ for all α, β) over some path-connected space, with connection identically zero. On this principal G-bundle with connection there is a canonical trivial orbifold group action, defined by taking $\gamma^g_\alpha \equiv 1$ for all $g \in \Gamma$ and all α. There is also a family of nontrivial orbifold group actions, defined by taking γ^g_α to be constant maps into G (i.e., $\gamma^g_\alpha = \gamma^g|_{U_\alpha}$), forming a representation of the orbifold group:

$$\gamma^{g_1 g_2} \;=\; \left(\gamma^{g_1}\right) \left(\gamma^{g_2}\right)$$

In other words, each set of $\{\gamma^g\}$ defining an orbifold group action defines an element of $\mathrm{Hom}(\Gamma, G)$. The reader can easily check that such γ^g yield a well-defined orbifold group action on the canonically trivial principal G-bundle with zero connection.

Now, we should be slightly careful. Not all of the elements of $\mathrm{Hom}(\Gamma, G)$ define distinct orbifold group actions on this special bundle with connection. Under a constant gauge transformation ϕ, the connection transforms as $A^\alpha \mapsto \phi A^\alpha \phi^{-1}$. As a result, given an orbifold group action defined by constant γ^g as

$$g^* A^\alpha \;=\; \left(\gamma^g\right) \left(A^\alpha\right) \left(\gamma^g\right)^{-1}$$

if we gauge-transform by constant ϕ we get

$$\phi\left(g^* A^\alpha\right)\phi^{-1} \;=\; \left(\gamma^g\right)\,\left(\phi A^\alpha \phi^{-1}\right)\,\left(\gamma^g\right)^{-1}$$

which can be rewritten as

$$g^* A^\alpha \;=\; \left(\phi^{-1}\gamma^g \phi\right)\,\left(A^\alpha\right)\,\left(\phi^{-1}\gamma^g\phi\right)^{-1}$$

In other words, a constant gauge transformation (on this special bundle with connection) will map an orbifold group action defined by $\{\gamma^g\}$ to an orbifold group action defined by $\{\phi^{-1}\gamma^g\phi\}$. Conversely, any two orbifold group actions that differ by conjugation by a constant map can be related by gauge transformation. Thus, on the canonical trivial bundle with trivial connection, distinct orbifold group actions are defined by elements of $\mathrm{Hom}(\Gamma, G)/G$, where modding out G is done by conjugation.

Thus, on canonically trivial bundles with zero connection, we find a family of orbifold group actions defined by $\mathrm{Hom}(\Gamma, G)/G$. This result is often used in discussions of heterotic orbifolds – for example, it can be found in[1] [7].

We should emphasize that the occurence of $\mathrm{Hom}(\Gamma, G)/G$ above for nonabelian G is much more restrictive than its occurrence for abelian G. For nonabelian G, we have found $\mathrm{Hom}(\Gamma, G)/G$ only for the special case of trivial principal G-bundles with zero connection. For abelian G, $\mathrm{Hom}(\Gamma, G)/G = \mathrm{Hom}(\Gamma, G)$ is ubiquitous – for abelian G, elements of this group define differences between orbifold group actions on any[2] principal G-bundle with connection.

5 Orbifold group actions on induced gerbes

Before we can understand orbifold group actions on heterotic B fields, we first need to work out the orbifold group actions on the Čech-de Rham cocycles associated to $\mathrm{Tr}\ F \wedge F$ and $\mathrm{Tr}\ R \wedge R$, as induced by orbifold group actions on the corresponding bundles with connection.

As mentioned previously, the Čech-de Rham cocycle description of $\mathrm{Tr}\ F \wedge F$ and $\mathrm{Tr}\ R \wedge R$ is describing the connection on an associated 2-gerbe. The orbifold group action on the gauge and tangent bundles will induce an orbifold group action on these associated 2-gerbes with connection. Now, orbifold group actions on 2-gerbes with connection were previously studied in [5], so we can borrow the results of that paper to write, in general:

$$g^* \omega_3^\alpha \;\;=\;\; \omega_3^\alpha \,+\, d\Lambda^{(2)}(g)^\alpha$$

[1]In that reference, the group $\mathrm{Hom}(\Gamma, G)/G$ is described in a rather obscure fashion. Specifically, it is described in terms of root and weight lattices, and only for the special case $\Gamma = \mathbf{Z}_n$.

[2]Assuming, as always, that the principal G-bundle with connection admits an action of the orbifold group Γ.

$$
\begin{aligned}
g^* \omega_2^{\alpha\beta} &= \omega_2^{\alpha\beta} \,+\, d\Lambda^{(1)}(g)^{\alpha\beta} \,+\, \Lambda^{(2)}(g)^{\alpha} \,-\, \Lambda^{(2)}(g)^{\beta} \\
g^* \omega_1^{\alpha\beta\gamma} &= \omega_1^{\alpha\beta\gamma} \,+\, d\log \nu^g_{\alpha\beta\gamma} \,+\, \Lambda^{(1)}(g)^{\alpha\beta} \,+\, \Lambda^{(1)}(g)^{\beta\gamma} \,+\, \Lambda^{(1)}(g)^{\gamma\alpha} \\
g^* h_{\alpha\beta\gamma\delta} &= \left(h_{\alpha\beta\gamma\delta} \right) \left(\nu^g_{\beta\gamma\delta} \right) \left(\nu^g_{\alpha\gamma\delta} \right)^{-1} \left(\nu^g_{\alpha\beta\delta} \right) \left(\nu^g_{\alpha\beta\gamma} \right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda^{(2)}(g_1 g_2)^{\alpha} &= \Lambda^{(2)}(g_2)^{\alpha} \,+\, g_2^* \Lambda^{(2)}(g_1)^{\alpha} \,+\, d\Lambda^{(3)}(g_1, g_2)^{\alpha} \\
\Lambda^{(1)}(g_1 g_2)^{\alpha\beta} &= \Lambda^{(1)}(g_2)^{\alpha\beta} \,+\, g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} \,-\, \Lambda^{(3)}(g_1, g_2)^{\alpha} \,+\, \Lambda^{(3)}(g_1, g_2)^{\beta} \\
&\quad - d\log \lambda^{g_1, g_2}_{\alpha\beta} \\
\Lambda^{(3)}(g_2, g_3)^{\alpha} \,+\, \Lambda^{(3)}(g_1, g_2 g_3)^{\alpha} &= g_3^* \Lambda^{(3)}(g_1, g_2)^{\alpha} \,+\, \Lambda^{(3)}(g_1 g_2, g_3)^{\alpha} \,+\, d\log \gamma^{g_1, g_2, g_3}_{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\nu^{g_1 g_2}_{\alpha\beta\gamma} &= \left(\nu^{g_2}_{\alpha\beta\gamma} \right) \left(g_2^* \nu^{g_1}_{\alpha\beta\gamma} \right) \left(\lambda^{g_1, g_2}_{\alpha\beta} \right) \left(\lambda^{g_1, g_2}_{\beta\gamma} \right) \left(\lambda^{g_1, g_2}_{\gamma\alpha} \right) \\
\left(\lambda^{g_1 g_2, g_3}_{\alpha\beta} \right) \left(g_3^* \lambda^{g_1, g_2}_{\alpha\beta} \right) &= \left(\lambda^{g_1, g_2 g_3}_{\alpha\beta} \right) \left(\lambda^{g_2, g_3}_{\alpha\beta} \right) \left(\gamma^{g_1, g_2, g_3}_{\alpha} \right) \left(\gamma^{g_1, g_2, g_3}_{\beta} \right)^{-1} \\
\left(\gamma^{g_1, g_2, g_3 g_4}_{\alpha} \right) \left(\gamma^{g_1 g_2, g_3, g_4}_{\alpha} \right) &= \left(\gamma^{g_1, g_2 g_3, g_4}_{\alpha} \right) \left(\gamma^{g_2, g_3, g_4}_{\alpha} \right) \left(g_4^* \gamma^{g_1, g_2, g_3}_{\alpha} \right)
\end{aligned}
$$

for some forms $\Lambda^{(2)}(g)^{\alpha}$, $\Lambda^{(1)}(g)^{\alpha\beta}$, $\Lambda^{(3)}(g_1, g_2)^{\alpha}$, $\nu^g_{\alpha\beta\gamma}$, $\lambda^{g_1, g_2}_{\alpha\beta}$, and $\gamma^{g_1, g_2, g_3}_{\alpha}$ which define the orbifold group action on the corresponding principal bundles with connection.

As a much simpler example, it is very straightforward to work out the orbifold group action induced on the 0-gerbe (determinant bundle) associated to some principal bundle with connection. Recall that the 0-gerbe with connection has curvature $\mathrm{Tr}\ F$, local connections $\mathrm{Tr}\ A^{\alpha}$, and transition functions $\det g_{\alpha\beta}$. Also recall that the orbifold group action on a principal G-bundle with connection is described by functions γ^g_{α}, where

$$
\begin{aligned}
g^* A^{\alpha} &= \left(\gamma^g_{\alpha} \right) A^{\alpha} \left(\gamma^g_{\alpha} \right)^{-1} \,+\, \left(\gamma^g_{\alpha} \right) d \left(\gamma^g_{\alpha} \right)^{-1} \\
g^* g_{\alpha\beta} &= \left(\gamma^g_{\alpha} \right) \left(g_{\alpha\beta} \right) \left(\gamma^g_{\beta} \right)^{-1} \\
\gamma^{g_1 g_2}_{\alpha} &= \left(g_2^* \gamma^{g_1}_{\alpha} \right) \left(\gamma^{g_2}_{\alpha} \right)
\end{aligned}
$$

From this description, it is easy to compute that

$$
\begin{aligned}
g^* \mathrm{Tr}\ F &= \mathrm{Tr}\ F \\
g^* \mathrm{Tr}\ A^{\alpha} &= \mathrm{Tr}\ A^{\alpha} \,+\, \mathrm{Tr}\ \left(\left(\gamma^g_{\alpha} \right) d \left(\gamma^g_{\alpha} \right)^{-1} \right) \\
&= \mathrm{Tr}\ A^{\alpha} \,+\, d\log \left(\det \gamma^g_{\alpha} \right) \\
\det \gamma^{g_1 g_2}_{\alpha} &= \left(\det \gamma^{g_2}_{\alpha} \right) \left(g_2^* \det \gamma^{g_1}_{\alpha} \right)
\end{aligned}
$$

so we see explicitly that the orbifold group action on a principal G-bundle with connection defines an orbifold group action on the associated 0-gerbe (determinant bundle) with connection.

6 Orbifold group actions on heterotic B-fields

Now that we have described heterotic B fields on local coordinate patches, and described the orbifold group actions induced on Chern-Simons forms by orbifold group actions on principal bundles with connection, we are finally ready to work out orbifold group actions on heterotic B fields.

First, recall that in the Green-Schwarz mechanism, gauge transformations of the bundle which induce

$$\omega_3^\alpha \mapsto \omega_3^\alpha - \mathrm{Tr}\ (d\Lambda \wedge dA^\alpha)$$

the B field transforms as

$$B^\alpha \mapsto B^\alpha + \mathrm{Tr}\ (\Lambda dA^\alpha)$$

(so that H remains invariant). From this fact and the fact that under the action of the orbifold group,

$$g^* \omega_3^\alpha = \omega_3^\alpha + d\Lambda^{(2)}(g)^\alpha$$

we see that, in general,

$$g^* B^\alpha = B^\alpha - \Lambda^{(2,F)}(g)^\alpha + \Lambda^{(2,R)}(g)^\alpha + d\Lambda^{(1,B)}(g)^\alpha \tag{10}$$

for some local one-forms $\Lambda^{(1,B)}(g)^\alpha$.

Also note that this implies that $g^* H = H$. In fact, we should have expected this – since H has no gauge transformations, any well-defined orbifold group action must map H back into precisely itself.

From the fact that

$$B^\alpha - B^\beta = dA^{\alpha\beta} - \omega_{2,F}^{\alpha\beta} + \omega_{2,R}^{\alpha\beta}$$

we can derive that

$$g^* A^{\alpha\beta} = A^{\alpha\beta} + \Lambda^{(1,B)}(g)^\alpha - \Lambda^{(1,B)}(g)^\beta + \Lambda^{(1,F)}(g)^{\alpha\beta} - \Lambda^{(1,R)}(g)^{\alpha\beta} + d\log \kappa_{\alpha\beta}^g \tag{11}$$

for some local function $\kappa_{\alpha\beta}^g$.

From the fact that

$$A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} = \omega_{1,F}^{\alpha\beta\gamma} - \omega_{1,R}^{\alpha\beta\gamma} + d\log h_{\alpha\beta\gamma}^B$$

we can derive that

$$g^* h_{\alpha\beta\gamma}^B = \left(h_{\alpha\beta\gamma}^B\right) \left(\nu_{\alpha\beta\gamma}^{Fg}\right)^{-1} \left(\nu_{\alpha\beta\gamma}^{Rg}\right) \left(\kappa_{\alpha\beta}^g\right) \left(\kappa_{\beta\gamma}^g\right) \left(\kappa_{\gamma\alpha}^g\right) \tag{12}$$

From expanding $(g_1 g_2)^* h_{\alpha\beta\gamma}^B$ in two different ways, we find that

$$\left(\lambda_{\alpha\beta}^{Fg_1,g_2}\right)^{-1} \left(\lambda_{\alpha\beta}^{Rg_1,g_2}\right) \left(\kappa_{\alpha\beta}^{g_1 g_2}\right) = \left(\kappa_{\alpha\beta}^{g_2}\right) \left(g_2^* \kappa_{\alpha\beta}^{g_1}\right) \left(h_\alpha^{g_1,g_2}\right) \left(h_\beta^{g_1,g_2}\right)^{-1} \tag{13}$$

for some local functions $h_\alpha^{g_1,g_2}$.

From writing $\kappa_{\alpha\beta}^{g_1 g_2 g_3}$ in two different ways, we find that

$$\left(\gamma_\alpha^{F g_1,g_2,g_3}\right) \left(\gamma_\alpha^{R g_1,g_2,g_3}\right)^{-1} \left(h_\alpha^{g_1 g_2,g_3}\right) \left(g_3^* h_\alpha^{g_1,g_2}\right) \;=\; \left(h_\alpha^{g_1,g_2 g_3}\right) \left(h_\alpha^{g_2,g_3}\right) \tag{14}$$

From expanding $(g_1 g_2)^* B^\alpha$ in two different ways, we find

$$-d\Lambda^{(3,F)}(g_1,g_2)^\alpha \;+\; d\Lambda^{(3,R)}(g_1,g_2)^\alpha \;+\; d\Lambda^{(1,B)}(g_1 g_2)^\alpha \;=\; d\Lambda^{(1,B)}(g_2)^\alpha \;+\; g_2^* d\Lambda^{(1,B)}(g_1)^\alpha \tag{15}$$

and from expanding $(g_1 g_2)^* A^{\alpha\beta}$ in two different ways, we find

$$\begin{aligned} \delta\left[\Lambda^{(1,B)}(g_1 g_2)^\alpha \;-\; \Lambda^{(3,F)}(g_1,g_2)^\alpha \right. \;&+\; \left. \Lambda^{(3,R)}(g_1,g_2)^\alpha \;+\; d\log h_\alpha^{g_1,g_2}\right] \\ &=\; \delta\left[\Lambda^{(1,B)}(g_2)^\alpha \;+\; g_2^* \Lambda^{(1,B)}(g_1)^\alpha\right] \end{aligned} \tag{16}$$

which we combine to conclude that

$$\Lambda^{(1,B)}(g_1 g_2)^\alpha - \Lambda^{(3,F)}(g_1,g_2)^\alpha + \Lambda^{(3,R)}(g_1,g_2)^\alpha + d\log h_\alpha^{g_1,g_2} \;=\; \Lambda^{(1,B)}(g_2)^\alpha + g_2^* \Lambda^{(1,B)}(g_1)^\alpha \tag{17}$$

To summarize, we have discovered that an orbifold group action on a heterotic B field is defined by

$$\begin{aligned} g^* H &= H \\ g^* B^\alpha &= B^\alpha \;-\; \Lambda^{(2,F)}(g)^\alpha \;+\; \Lambda^{(2,R)}(g)^\alpha \;+\; d\Lambda^{(1,B)}(g)^\alpha \\ g^* A^{\alpha\beta} &= A^{\alpha\beta} \;+\; \Lambda^{(1,B)}(g)^\alpha \;-\; \Lambda^{(1,B)}(g)^\beta \\ &\quad + \Lambda^{(1,F)}(g)^{\alpha\beta} \;-\; \Lambda^{(1,R)}(g)^{\alpha\beta} \;+\; d\log \kappa_{\alpha\beta}^g \\ g^* h_{\alpha\beta\gamma}^B &= \left(h_{\alpha\beta\gamma}^B\right) \left(\nu_{\alpha\beta\gamma}^{Fg}\right)^{-1} \left(\nu_{\alpha\beta\gamma}^{Rg}\right) \left(\kappa_{\alpha\beta}^g\right) \left(\kappa_{\beta\gamma}^g\right) \left(\kappa_{\gamma\alpha}^g\right) \\[1ex] \left(\lambda_{\alpha\beta}^{F g_1,g_2}\right)^{-1} \left(\lambda_{\alpha\beta}^{R g_1,g_2}\right) \left(\kappa_{\alpha\beta}^{g_1 g_2}\right) &= \left(\kappa_{\alpha\beta}^{g_2}\right) \left(g_2^* \kappa_{\alpha\beta}^{g_1}\right) \left(h_\alpha^{g_1,g_2}\right) \left(h_\beta^{g_1,g_2}\right)^{-1} \\ \left(h_\alpha^{g_1 g_2,g_3}\right) \left(g_3^* h_\alpha^{g_1,g_2}\right) &= \left(h_\alpha^{g_1,g_2 g_3}\right) \left(h_\alpha^{g_2,g_3}\right) \left(\gamma_\alpha^{F g_1,g_2,g_3}\right)^{-1} \left(\gamma_\alpha^{R g_1,g_2,g_3}\right) \\ \Lambda^{(1,B)}(g_1 g_2)^\alpha \;+\; d\log h_\alpha^{g_1,g_2} &= \Lambda^{(3,F)}(g_1,g_2)^\alpha \;-\; \Lambda^{(3,R)}(g_1,g_2)^\alpha \;+\; \Lambda^{(1,B)}(g_2)^\alpha \\ &\quad + g_2^* \Lambda^{(1,B)}(g_1)^\alpha \end{aligned}$$

for some $\Lambda^{(1,B)}(g)^\alpha$, $\kappa_{\alpha\beta}^g$, and $h_\alpha^{g_1,g_2}$ introduced to define the orbifold group action on the heterotic B field. Note that this is the same set of data needed to define an orbifold group action on a B field for the case $dH = 0$ [2, 3, 4]; the difference in the present case is that the orbifold group action is warped by the interaction with the gauge and tangent bundles.

7 Differences between orbifold group actions

In [2, 3, 4], the group $H^2(\Gamma, U(1))$ was recovered when describing the differences between orbifold group actions on B fields such that $dH = 0$. With that in mind, we shall now examine the differences between orbifold group actions on heterotic B fields.

Assume the orbifold group actions on the gauge and tangent bundles are fixed. Let the data defining the two orbifold group actions on the heterotic B field be distinguished by an overline. Define

$$\begin{aligned} T^g_{\alpha\beta} &= \frac{\kappa^g_{\alpha\beta}}{\overline{\kappa}^g_{\alpha\beta}} \\ A(g)^\alpha &= \overline{\Lambda}^{(1,B)}(g)^\alpha \, - \, \Lambda^{(1,B)}(g)^\alpha \\ \omega^{g_1,g_2}_\alpha &= \frac{h^{g_1,g_2}_\alpha}{\overline{h}^{g_1,g_2}_\alpha} \end{aligned}$$

From the expressions

$$\begin{aligned} g^* B^\alpha &= B^\alpha \, - \, \Lambda^{(2,F)}(g)^\alpha \, + \, \Lambda^{(2,R)}(g)^\alpha \, + \, d\Lambda^{(1,B)}(g)^\alpha \\ &= B^\alpha \, - \, \Lambda^{(2,F)}(g)^\alpha \, + \, \Lambda^{(2,R)}(g)^\alpha \, + \, d\overline{\Lambda}^{(1,B)}(g)^\alpha \end{aligned}$$

we see that

$$dA(g)^\alpha \, = \, 0 \tag{18}$$

From writing $g^* A^{\alpha\beta}$ in two different ways, we find that

$$A(g)^\alpha \, - \, A(g)^\beta \, = \, d \log T^g_{\alpha\beta} \tag{19}$$

From writing $g^* h^B_{\alpha\beta\gamma}$ in two different ways, we find that

$$\left(T^g_{\alpha\beta} \right) \left(T^g_{\beta\gamma} \right) \left(T^g_{\gamma\alpha} \right) \, = \, 1 \tag{20}$$

From the equations above, we see that the $T^g_{\alpha\beta}$ are transition functions for a principal $U(1)$ bundle with connection defined by $A(g)^\alpha$, and that that connection is flat.

By dividing the expressions for $\kappa^{g_1 g_2}_{\alpha\beta}$ and $\overline{\kappa}^{g_1,g_2}_{\alpha\beta}$, we find that

$$T^{g_1 g_2}_{\alpha\beta} \, = \, \left(T^{g_2}_{\alpha\beta} \right) \left(g_2^* T^{g_1}_{\alpha\beta} \right) \left(\omega^{g_1,g_2}_\alpha \right) \left(\omega^{g_1,g_2}_\beta \right)^{-1} \tag{21}$$

From subtracting the expressions for $\Lambda^{(1,B)}(g)^\alpha$ and $\overline{\Lambda}^{(1,B)}(g)^\alpha$, we find that

$$A(g_1 g_2)^\alpha \, - \, d \log \omega^{g_1,g_2}_\alpha \, = \, A(g_2)^\alpha \, + \, g_2^* A(g_1)^\alpha \tag{22}$$

These two expressions tell us that the $\omega_\alpha^{g_1,g_2}$ define connection-preserving bundle isomorphisms

$$\omega^{g_1,g_2} : T^{g_2} \otimes g_2^* T^{g_1} \longrightarrow T^{g_1 g_2}$$

Finally, by dividing the expressions

$$\begin{aligned}
\left(\gamma_\alpha^{F g_1,g_2,g_3}\right) \left(\gamma_\alpha^{R g_1,g_2,g_3}\right)^{-1} \left(h_\alpha^{g_1 g_2,g_3}\right) \left(g_3^* h_\alpha^{g_1,g_2}\right) &= \left(h_\alpha^{g_1,g_2 g_3}\right) \left(h_\alpha^{g_2,g_3}\right) \\
\left(\gamma_\alpha^{F g_1,g_2,g_3}\right) \left(\gamma_\alpha^{R g_1,g_2,g_3}\right)^{-1} \left(\overline{h}_\alpha^{g_1 g_2,g_3}\right) \left(g_3^* \overline{h}_\alpha^{g_1,g_2}\right) &= \left(\overline{h}_\alpha^{g_1,g_2 g_3}\right) \left(\overline{h}_\alpha^{g_2,g_3}\right)
\end{aligned}$$

we find that

$$\left(\omega_\alpha^{g_1 g_2,g_3}\right) \left(g_3^* \omega_\alpha^{g_1,g_2}\right) = \left(\omega_\alpha^{g_1,g_2 g_3}\right) \left(\omega_\alpha^{g_2,g_3}\right) \tag{23}$$

This means that the connection-preserving bundle morphisms ω^{g_1,g_2} must make the following diagram commute:

$$\begin{array}{ccc}
T^{g_3} \otimes g_3^* \left(T^{g_2} \otimes g_2^* T^{g_1} \right) & \stackrel{\omega^{g_1,g_2}}{\longrightarrow} & T^{g_3} \otimes g_3^* T^{g_1 g_2} \\
\omega^{g_2,g_3} \downarrow & & \downarrow \omega^{g_1 g_2,g_3} \\
T^{g_2 g_3} \otimes (g_2 g_3)^* T^{g_1} & \stackrel{\omega^{g_1,g_2 g_3}}{\longrightarrow} & T^{g_1 g_2 g_3}
\end{array} \tag{24}$$

So far we have recovered the fact that the difference between two orbifold group actions on heterotic B fields (with fixed orbifold group actions on the gauge and tangent bundles) is defined by the same data as for B fields such that $dH = 0$ [2, 3, 4]: namely, pairs (T^g, ω^{g_1,g_2}) of bundles T^g with flat connection and connecting morphisms ω^{g_1,g_2} making diagram (24) commute.

Also, orbifold group actions on B fields are subject to the same equivalences as in [2, 3, 4]. If $\kappa_g : T^g \rightarrow T'^g$ is a connection-preserving isomorphism of principal G-bundles, then we can replace the data (T^g, ω^{g_1,g_2}) with the data $\left(T'^g, \kappa_{g_1 g_2} \circ \omega^{g_1,g_2} \circ \left(\kappa_{g_2} \otimes g_2^* \kappa_{g_1}\right)^{-1}\right)$.

Since the differences between orbifold group actions on heterotic B fields are defined by precisely the same data as for type II B fields [2, 3, 4], we recover the group $H^2(\Gamma, U(1))$ as well as the twisted-sector phases of [1] in precisely the same fashion as [2, 3, 4].

8 Conclusions

In this paper we have outlined a purely mathematical understanding of discrete torsion for heterotic B fields, as opposed to type II B fields, thereby filling a gap present in the earlier work [2, 3, 4]. Specifically, after working out a gerbe-like description of heterotic B fields, and after discussing orbifold group actions on principal G-bundles with connection for non-abelian G, we use a self-consistent bootstrap (in the style of [4]) to construct orbifold group

actions on B fields. Discrete torsion arises in the same fashion as in [2, 3, 4], namely in terms of the difference between orbifold group actions.

As in [2, 3, 4], the results in this paper do not assume that the orbifold group acts freely. Also as in [2, 3, 4], we do not assume the heterotic B field has vanishing curvature (though, as in [2, 3, 4], one needs to check that orbifold group actions on a given field configuration actually exist before attempting to formally classify them).

Finally, as in [2, 3, 4], our analysis does not assume any features of string theory. As in [2, 3, 4], discrete torsion can be understood in a purely mathematical framework, without any reference to string theory. In other words, there is nothing "inherently stringy" about discrete torsion.

One loose end we have had difficulty tying up involves the level-matching conditions of heterotic orbifolds. We strongly suspect that satisfying the level-matching conditions is equivalent to the statement that the orbifold group actions on the gauge and tangent bundles are consistent with the orbifold group action on the heterotic B field. In other words, we suspect the level-matching condition is equivalent to demanding that the orbifold group action on the heterotic B field be well-defined. Unfortunately, we have not yet been able to show this rigorously.

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